On removal of one undetected paradox in foundations of probability theory

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Abstract

The paper points out a logical contradiction that arises in the modern approach to teaching the foundations of the probability theory. This contradiction is of the following form: $\omega = \{\omega\}$, where ω is an arbitrary elementary outcome. In modern mathematics the formation of expressions of the form $x = \{x\}$ is unacceptable. Here $\{x\}$ is a unit set, i.e., a set whose sole element is some element x of an arbitrary nature. This paradox $\omega = \{\omega\}$ is a consequence of the fact that random events and between them also elementary outcomes ω are considered as some subsets in the set of all elementary outcomes. It is shown how this contradiction can be easily eliminated by applying the axiomatic method. It is based on the introduction of two simple axioms that impose restrictions on elementary outcomes and on the class of random events under consideration. In addition, from these two axioms on the basis of strictly logical reasoning the authors derived the representation of an arbitrary random event in the form of a sum of elementary outcomes favorable to it. Moreover, another non-axiomatic approach to the elimination of this paradox is proposed. In the final part of the paper other variants of the second of the introduced axioms are considered as well.

Keywords 1

Random event, elementary outcomes, principle of extension, contradiction, paradox, set, axiom, algebra of events, favorable, probability

1. Introduction

Let x be some element of arbitrary nature. Then $\{x\}$ is a so-called unit set, i.e., the set whose sole member is x. In modern mathematics the equality $x = \{x\}$ is considered as a gross mistake, since it is a violation of the principle of set extension [1]: two sets are equal if and only if they have the same elements. It is necessary to distinguish between an object x and a set $\{x\}$, the only element of which is this object, since the element x and the set $\{x\}$ containing it are multilevel objects. It is true that $x \in \{x\}$, but $x \neq \{x\}$. Thus, a set-theoretic contradiction $x = \{x\}$ results from a confusion between the membership relation and the relation of equality.

The modern study of set theory was initiated by the German mathematician Georg Cantor in the 1870s. In particular, Georg Cantor is commonly considered as the founder of set theory. The starting points of Cantor's set theory are based not on strict definitions and axioms, but on explanations. In this connection, this theory is also called naive set theory. Pretty soon, at the turn of the nineteenth and twentieth centuries, it became clear that in Cantor's set theory, contradictions arise. These contradictions are modestly called antinomies (paradoxes) of set theory. In particular, Russell's paradox, published in 1903, made a great impression on his contemporaries. In this paradox the existence of a set of all sets that are not their own element turns out to be contradictory. The emergence of such contradictions was associated with an overly broad point of view on the concept of a set and the methods of its formation.

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It became clear that there was a need to restrict Cantor's set theory. In order to avoid such contradictions, various axiomatic set theories have been developed (for example, the Zermelo-Fraenkel system of axioms [2]; nowadays it has become the most widespread). In these axiomatics some methods of constructing sets are distinguished which are considered admissible. The introduction of axiomatic set theories made it possible to discard "sets" whose definitions turned out to be internally contradictory. In an informal language the imposed restrictions on the methods of new sets generation require, in particular, that new sets can be generated only from the existing sets. Thus, among the sets a natural hierarchy is generated from simple sets to more complex ones. In particular, in order to control the classical Russell's paradox the axioms in axiomatic set theories are chosen so that sets that are their own elements cannot be considered. Now pay attention to the fact that there is a set on the right-hand side in the equality $x = \{x\}$. Therefore, there must also be a set on the left-hand side. It means that x is precisely the set that has itself as its own element. But as indicated above in formal set theory such a construction is prohibited: no one set should be its own element. The introduction of such "sets" into circulation leads to irreparable contradictions.

In the foundations of probability theory this contradiction $x = \{x\}$ appears in the form $\omega = \{\omega\}$ where ω is an arbitrary elementary outcome. This is due to the fact that random events and, in particular, elementary outcomes as well are formally identified with some subsets in the set of elementary outcomes [3, 4]. There should be no contradictions in mathematics and its foundations. If any contradiction is found, then it should be eliminated. Since the indicated contradiction is caused partly by the confusion of the set-theoretic language and designations with the language and designations of the theory of probability, the authors tried to distinguish them as clearly as possible where necessary.

The authors use the following more or less standard notation and some convenient abbreviations:

1) The term event-set, defined below, is abbreviated to coset or somon;
2) Impossible and certain events are designated V and U accordingly;
3) The notation $A \to B$ means that a random event A favors a random event B or entails it. In this case, a random event A is called a subevent of a random event B, and B is called an over-event. If $A \neq V$ and $A \neq B$, then the random event A is called its proper subevent;
4) The expression AB means the product of random events A and B. If $AB = V$, then random events A and B are called inconsistent;
5) If we have an indexed family of pairwise incompatible random events $S_\alpha, \alpha \in I$, then their sum $\sum_{\alpha \in I} S_\alpha$ is, by definition, a random event that occurs when at least one random event from this family occurs. In particular, the notation $A + B$ of the sum of random events A and B implies that random events A and B are inconsistent. If it is written $\sum S_\alpha$, it means that the "summation" is carried out over all admissible values α. If the family S_α consists of one random event S, then $\sum_{\alpha \in I} S_\alpha := S$. If random events S_α are not necessarily pairwise inconsistent, then instead of their sum we write $\bigvee_{\alpha \in I} S_\alpha$;
6) The difference between random events A and B is denoted as $A \backslash B$;
7) The opposite to a random event S is designated as $\overline{S}$.
8) The expression $A \nrightarrow B$ means that a random event A does not favor a random event B, i.e., the occurrence of the event A does not necessarily mean the occurrence of the event B.

The axiomatic method helps the authors demonstrate how to avoid this paradox $\omega = \{\omega\}$. The paper will also clarify where and how this paradox arises. Modern textbooks on the probability theory introduce elementary outcomes ω that are informally defined at best as the outcomes of an experiment that are characterized in great detail [4]. Some textbooks additionally state that elementary outcomes should form a complete group of pairwise incompatible events. In relatively rare cases it is also indicated that the elementary outcomes must be "indecomposable" [4]. Nevertheless, no general strict logical conclusions are made from such definitions. Then the set of all elementary outcomes $\Omega = \{\omega : \omega \; is \; \text{elementary outcome}\}$ is introduced, and random events are defined as some subsets in Ω [5] or are identified with them [4]. This indicated scheme is demonstrated with a few simple examples that should confirm the correct view of random events as subsets in Ω. Thus, teaching in this case is carried out in the spirit of Babylonian mathematics [6]. It is known that among the cuneiform tablets found in Mesopotamia, a sufficient number (already more than 400) tablets related to mathematics were found. Nevertheless, none of these tablets has been found that would contain proofs of mathematical statements till present days. On specific examples, only schemes and algorithms for the solution are

presented, and there is no analysis of the solution. A detailed examination of these solutions shows that, apparently, the Babylonians had some kind of developed general mathematical theory. However, there was no systematic evidence-based approach. A student, it seems, was only required to analyze a number of examples and in this way understand for himself a general course of solving a certain class of problems.

2. Axiomatic approach

Get to a rigorous construction of the foundations of the Probability Theory, which does not lead to a contradiction $\omega = \{\omega\}$. So, we have some trial and random events associated with it. As a rule, they are not interested in all possible or conceivable random events, but only in some of their set, which is natural to consider the algebra of events A [4]. Moreover, among all events there is one more class of events different from the impossible one called elementary outcomes. In the general case elementary outcomes are not required to enter the algebra of events. But we first consider them to be a part of the algebra of events. Regarding the algebra of events A and the set of all elementary outcomes Ω, the following initial unsubstantiated statements or axioms are assumed to be fulfilled:

Axiom 1. $\sum \omega = U$, *where the summation is over all elementary outcomes* $\omega \in \Omega$.

Axiom 2. *For any random event* $S \in A$ *different from the impossible one and for any elementary outcome* $\omega \in \Omega$ *the relation* $S \to \omega \;\Rightarrow\; S = \omega$ *holds.*

Remark 1. Axiom 1 is simply a formal notation of the requirement for elementary outcomes to form a complete group of pairwise incompatible events. Moreover, a set or space of all elementary outcomes Ω is not assumed to be finite. And in the case of its infinity it does not have to be countably infinite.

Remark 2. Axiom 2 implicitly states that any elementary outcome is an indecomposable random event A with respect to the algebra of events in the sense that it cannot be represented as a sum of at least two pairwise incompatible random events A different from the impossible one. Indeed, let $\omega = \sum S_\alpha$. Then $S_\alpha \to \omega$ for all α and, therefore, according to Axiom 2, all $S_\alpha = \omega$ that contradicts the pairwise incompatibility of random events S_α. And vice versa, if some random event B has at least one proper subevent A, i.e., $A \to B$ with $A \neq B$ and $A \neq V$, then $B = A + (B \backslash A)$, where the random event $(B \backslash A) \neq V$. Hence, the random event B turns out to be decomposable in this case.

Remark 3. The authors will consider other alternatives of Axiom 2 in the last section.

The following Theorem 1 states that every random event from the algebra of events, different from the impossible event, is decomposable into the sum of elementary outcomes that favor it.

Theorem 1. *Let* $S \in A$ *and* $S \neq V$, *then*

$$S = \sum_{\omega \,:\, \omega \to S} \omega. \qquad (1)$$

Proof: Let B be a random event on the right-hand side of (1). Prove that $S = B$. It is enough to check that $B \to S$ and $S \to B$.

a) Check that $B \to S$. Let a random event B occur as a result of an experiment. Consequently, one of the elementary outcomes ω has come such that $\omega \to S$. It means that a random event S has occurred. Thus, the occurrence of a random event B entailed the occurrence of a random event S, i.e., $B \to S$.

b) Now let us check that $S \to B$. Let a random event S occur. As a result of the test according to Axiom 1 one of the elementary outcomes should occur. Designate it as ω'. It means that a random event $S\omega'$ has occurred. Obviously that $S\omega' \to S$ and $S\omega' \to \omega'$. Since S and ω' are included in the algebra of events, the event $S\omega' \in A$ as well. But then, according to Axiom 2, we have that $S\omega' \to \omega' \;\Rightarrow\; S\omega' = \omega'$. Therefore, $\omega' = S\omega' \to S$, i.e., $\omega' \to S$. Thus, when a random event S occurs, an elementary outcome ω'occurs such that $\omega' \to S$. It means the occurrence of a random event B. So, $S \to B$ as required to prove. This concludes the proof of Theorem 1.

Remark 4. Another proof of Theorem 1 can be presented. It may look somewhat more formal. It uses the following properties of operations on random events:

a) $SU = S$ and b) $S(\sum S_\alpha) = \sum(SS_\alpha)$.

The first of these properties, of course, is obvious, and the second one is true, since, as is easy to verify $S(\sum S_\alpha) \rightarrow \sum(SS_\alpha)$ and vice versa $\sum(SS_\alpha) \rightarrow S(\sum S_\alpha)$.

Let the set $C := \{ \omega \in \Omega : S\omega \neq V\}$. Then, using property a), Axiom 1, and property b), we have

$$S = SU = S\sum_{\omega : \omega \in \Omega}\omega = \sum_{\omega : \omega \in \Omega} S\omega = \sum_{\omega : \omega \in C} S\omega \qquad (2)$$

Then as in the previous proof of Theorem 1 we have $S\omega \rightarrow S$ and $S\omega \rightarrow \omega$. And since $S\omega \neq V$ for $\omega \in C$, according to Axiom 2 we have $S\omega \rightarrow \omega \Rightarrow S\omega = \omega$. So we showed that $\omega \in C \Rightarrow S\omega = \omega$ and that $\omega \in C \Rightarrow \omega \rightarrow S$. At the same time $\omega \rightarrow S \Rightarrow S\omega \neq V$. It means that the set $C = \{ \omega \in \Omega : \omega \rightarrow S\}$. Therefore, we finally get from equalities (2) that $S = \sum_{\omega : \omega \in C} S\omega = \sum_{\omega : \omega \rightarrow S}\omega$, as required to establish.

To each random event $S \in A$ we associate a subset $\tilde{S}$ in the set of elementary outcomes Ω putting $\tilde{S}$: $= \{ \omega \in \Omega : \omega \rightarrow S \}$. Thus, we have that $\omega \in \tilde{S} \Leftrightarrow \omega \rightarrow S$.

Lemma 1. *Let random events $A, B \in A$. Then $A \rightarrow B \Leftrightarrow \tilde{A} \subseteq \tilde{B}$.*

Proof: Let $A, B \neq V$. From formula (1) it follows that a random event from the algebra of events, different from the impossible one, occurs if and only if one of the elementary outcomes that favor it occurs. At the same time according to Theorem 1 random events A and B are represented as $A = \sum_{\omega : \omega \rightarrow A}\omega$ and $B = \sum_{\omega : \omega \rightarrow B}\omega$. Therefore, we can write the following chain of equivalences:

$$(A \rightarrow B) \Leftrightarrow (\forall\omega \in \Omega,\ \omega \rightarrow A \Rightarrow \omega \rightarrow B) \Leftrightarrow (\forall\omega \in \Omega,\ \omega \in \tilde{A} \Rightarrow \omega \in \tilde{B}) \Leftrightarrow (\tilde{A} \subseteq \tilde{B}).$$

The lemma has been proved.

Consequence 1. *If $A, B \in A$, then $A = B \Leftrightarrow \tilde{A} = \tilde{B}$.*

Proof: The proof follows immediately from the fact that, firstly, $A = B \Leftrightarrow A \rightarrow B$ and $B \rightarrow A$. Secondly, $\tilde{A} = \tilde{B} \Leftrightarrow \tilde{A} \subseteq \tilde{B}$ and $\tilde{B} \subseteq \tilde{A}$.

Remark 5. Note that the equality A=B is the equality of two random events, and the equality $\tilde{A} = \tilde{B}$ is the equality of two subsets in the space of elementary outcomes Ω.

Let us call $\tilde{S}$ as the event-set to the random event S and for brevity we will call it a coset or somon. The following Theorem 2 shows that operations on random events correspond to set-theoretic operations on their somons..

Theorem 2. *Let random events $A, B, S, S_\alpha \in A$. The following equalities are true*

1) $\tilde{V} = \emptyset$, and $\tilde{U} = \Omega$,
2) $\widetilde{AB} = \tilde{A} \cap \tilde{B}$,
3) $\widetilde{A \vee B} = \tilde{A} \cup \tilde{B}$ and more generally $\widetilde{\vee S_\alpha} = \cup\ \tilde{S}_\alpha$, $\alpha \in I$,
4) $\tilde{\tilde{S}} = (\tilde{S})^c$, where $(\tilde{S})^c = \Omega - \tilde{S}$ is the absolute complement of the set $\tilde{S}$ relative to Ω.

Proof: Check, for example, that $\widetilde{\vee S_\alpha} = \cup \tilde{S}_\alpha$, $\alpha \in I$. The remaining equalities are established in a similar way. We have the following chain of equivalences

$$\omega \in \widetilde{\vee S_\alpha} \Leftrightarrow \omega \rightarrow \vee S_\alpha \Leftrightarrow \exists\beta \in I:\ \omega \rightarrow S_\beta \Leftrightarrow \exists\beta \in I:\ \omega \in \tilde{S}_\beta \Leftrightarrow \omega \in \cup \tilde{S}_\alpha.$$

Thus, $\omega \in \widetilde{\vee S_\alpha} \Leftrightarrow \omega \in \cup \tilde{S}_\alpha$, i.e., the sets $\widetilde{\vee S_\alpha}$ and $\cup \tilde{S}_\alpha$ consist of the same elements and, according to the principle of extension, these sets are equal. The theorem has been proved.

Consequence 2. First, it follows from equalities 1) and 2) that if two events are inconsistent, then their somons do not intersect. So, according to equality 3), the sum of pairwise incompatible random events corresponds to the union of their pairwise disjoint somons. Secondly, the somons also form the algebra of subsets $\tilde{A}$ in the set of all elementary outcomes Ω. Of course, if A is a σ-algebra, then $\tilde{A}$ is a σ-algebra as well. Moreover, the correspondence between events from A and their somons from $\tilde{A}$ is an isomorphism between algebras (σ-algebras) A and $\tilde{A}$.

In the modern teaching of Probability Theory there is an identification of random events S and their somons $\tilde{S}$. The set $\tilde{S}$ consisting of elementary outcomes favorable to a random event S is called a random event and further S and $\tilde{S}$ are not distinguished, i.e., formally it turns out that $S \equiv \tilde{S}$. Definitely, such an identification is not entirely legitimate, since objects S and $\tilde{S}$ are of different nature: S is a random event, i.e., some qualitative or quantitative characteristic of the test result, and according to Theorem 1 is the sum of elementary outcomes favorable to it, and $\tilde{S}$ is a subset in the set of elementary outcomes. Moreover, since $\widetilde{\omega} = \{\omega\}$, in this particular case the identification $\omega \equiv \widetilde{\omega}$ means equality, $\omega = \{\omega\}$, and hence leads to the mistake of not distinguishing between an element and a unit set containing this element. So, if we want to avoid the mistake $\omega = \{\omega\}$, then we must adhere to a strict distinction between a random event S and its somon $\tilde{S}$ and avoid identifying them.

A few words about the axiomatic definition of probability. According to this definition we have the probability space (Ω, A', Pr), where Pr is a probability measure defined on the σ – algebra A' of subsets Ω satisfying the standard requirements [4, 5]. According to the approach outlined in the paper the elements A' are somons, not random events. In this case, the probability $P(S)$ of a random event S should be defined by the formula $P(S) := Pr(\tilde{S})$ for those random events for which $\tilde{S} \in A'$. A similar remark should be made regarding the definition of geometric probability. Here $\tilde{S} \subseteq R^m$ and precisely for it the m-dimensional volume (measure) $v(\tilde{S})$ is determined. Therefore, it is necessary to put $P(S) := v(\tilde{S})/v(\Omega)$.

3. Another approach different from axiomatic one

Thus, if it is necessary to avoid the mistake $\omega = \{\omega\}$, then we must adhere to a strict distinction between a random event S and its somon $\widetilde{S}$ and avoid their identification. However, another approach to eliminating this paradox is also possible. Specifically, call the somon $\tilde{S}$ a random event that occurs as a result of a test if and only if one (and only one, according to Axiom 1) occurs from the elementary outcomes that form $\tilde{S}$, i.e., elementary outcomes favorable to a random event S. Since, according to formula (1), a random event S also occurs if and only if one of the elementary outcomes favorable to it occurs, then random events S and $\tilde{S}$ are equal, i.e., $S = \tilde{S}$. Consequently, now the equality $S = \tilde{S}$ is the equality of two random events, and not the result of their identification. Thus, from this point of view, somon $\tilde{S}$ can be viewed as a convenient designation for a random event S. Accordingly, equality $\omega = \{\omega\}$ should now also be understood not in the set-theoretic sense as the equality of an element ω and a unit set $\{\omega\}$, but exclusively as the equality of two random events: a random event ω and a correctly understood random event $\widetilde{\omega} = \{\omega\}$.

Definitely, this approach can be confusing. Moreover, as the simplest examples of finding the classical probability show, the space of elementary outcomes can be determined not uniquely. So in this case a random event may have several completely different somons, which should also be equated with each other as equal events.

4. Other forms of Axiom 2

A simple analysis of the proofs of Theorem 1 shows that to obtain formula (1) we can replace Axiom 2 with the following Axiom 3:

Axiom 3. *For any random event $S \in A$ and any elementary outcome $\omega \in \Omega$ the following relation holds:*

$$S\omega = \begin{cases} \omega, \text{ if } \omega \to S; \\ V, \text{ if } \omega \not\to S. \end{cases}$$

In this case there is no need to require the inclusion of elementary outcomes in the algebra of events. Formula (1) can be easily deduced from Axiom 1 and Axiom 3 as follows:

$$S = SU = S\sum \omega = \sum S\omega = \sum_{\omega\,:\,\omega \to S} \omega.$$

Regardless of Axioms 1-3, the following theorem holds:

Theorem 3. *For any random events A and B the following relation holds:*

$$A \to B \;\Rightarrow\; AB = A.$$

Proof: To establish the equality $AB = A$ it is sufficient to demonstrate that $AB \to A$ and $A \to AB$. But $AB \to A$ always. On the other hand, if $A \to B$, then $A \to AB$. The theorem has been proved.

Theorem 3 shows that the first equality $S\omega = \omega$, if $\omega \to S$, in Axiom 3 always holds automatically. Therefore, Axiom 3 can be reformulated in a simpler form as Axiom 4.

Axiom 4. *For any random event $S \in A$ and any elementary outcome $\omega \in \Omega$ the following relation $\omega \not\to S \;\Rightarrow\; S\omega = V$ holds.*

Theorem 1 still remains true as a logical consequence of Axiom 1 and Axiom 4.

Also, Axiom 2 can be replaced by the following Axiom 5:

Axiom 5. *For any random event $S \in A$ and any elementary outcome $\omega \in \Omega$ a random event $S\omega$* is either V or ω.

Formula (1) is derived from Axiom 1 and Axiom 5 as follows:

$$S = SU = S\sum\omega = \sum S\omega = \sum_{\omega\,:\,S\omega\neq V}\omega.$$

So, $S = \sum_{\omega\,:\,S\omega\neq V}\omega$. But if the elementary outcome ω is included in the last sum, then it necessarily favors the random event S. On the other hand, if $\omega \to S$, then according to Theorem 3 we obtain $S\omega = \omega \neq V$. Therefore, $S = \sum_{\omega\,:\,S\omega\neq V}\omega = \sum_{\omega\,:\,\omega\to S}\omega$ which was required to be demonstrated.

Theorem 4. *Suppose that Axiom 1 and the statement of Theorem 1 hold. This means in particular that any random event $S \in A$ different from the impossible one is represented in the form of sum (1): $S = \sum_{\omega\,:\,\omega\to S}\omega$. Then the statements of Axioms 2, 3, 4 and 5 hold.*

Proof: According to Theorem 3 Axioms 3 and 4 are equivalent.

a) Check the validity of Axiom 5. Let ω' be an arbitrary elementary outcome. Then

$$S\omega' = \left(\sum_{\omega\,:\,\omega\to S}\omega\right)\omega' = \sum_{\omega\,:\,\omega\to S}(\omega\,\omega').$$

Since by virtue of Axiom 1 all elementary outcomes are pairwise inconsistent, then $\omega\omega' = V$, if $\omega \neq \omega'$, or $\omega\omega' = \omega'$, if $\omega = \omega'$. So, $S\omega'$ is either V or ω'. It means that Axiom 5 holds.

b) Check the validity of Axiom 4. If an elementary outcome $\omega' \not\to S$, then it is not included in the sum $\sum_{\omega\,:\,\omega\to S}\omega = S$. Since the elementary outcomes are pairwise inconsistent, then $\omega\omega' = V$ for any elementary outcome from this sum. Hence,

$$S\omega' = \left(\sum_{\omega\,:\,\omega\to S}\omega\right)\omega' = \sum_{\omega\,:\,\omega\to S}(\omega\,\omega') = V.$$

So, $\omega' \not\to S \Rightarrow S\omega' = V$, and hence, Axiom 4 holds.

c) Check the validity of Axiom 2. Let $S \in A$ *and* $S \neq V$. Suppose that $S \to \omega'$, where ω' is an arbitrary elementary outcome. According to Theorem 3, the equality $S\omega' = S$ holds. Then

$$S = S\omega' = \left(\sum_{\omega\,:\,\omega\to S}\omega\right)\omega' = \sum_{\omega\,:\,\omega\to S}(\omega\,\omega').$$

Since the elementary outcomes are pairwise inconsistent, the last sum can contain no more than one term ω'. As $S \neq V$, then one term must necessarily be. It means that $S = \sum_{\omega\,:\,\omega\to S}(\omega\,\omega') = \omega'$. Thus, $S \to \omega' \Rightarrow S = \omega'$, and hence, Axiom 2 holds.

Theorem 4 has been completely proved.

From the results of this section we immediately get the following theorem:

Theorem 5. *The following pairs of statements are equivalent: a) Axiom 1 and Axiom 2, b) Axiom 1 and Axiom 3, c) Axiom 1 and Axiom 4, d) Axiom 1 and Axiom 5, e) Axiom 1 and Theorem 1, i.e., if one pair of these statements holds, then all the others hold as well.*

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